NAME:

Solutions to Math 150 Practice Exam 3.2

Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. Find a simple expression for $\int \frac{4x^4-6x^2}{x} dx$ [10 pts]

Solution: $\int \frac{4x^4-6x^2}{x} dx = \int (4x^3 - 6x)\, dx = x^4 - 3x^2 + C$

2. Find a simple expression for $\int \frac{\sin\theta - 1}{\cos^2\theta} d\theta$ [10 pts]

Solution: $\int \frac{\sin\theta - 1}{\cos^2\theta} d\theta = \int (\sec\theta\tan\theta - \sec^2\theta) d\theta = \sec\theta - \tan\theta + C$

3. Use geometry to evaluate $\int_{-1}^{3} \sqrt{4-(x+1)^2} dx$ [10 pts]

Solution: Let $y = \sqrt{4-(x+1)^2}$. By squaring both sides of the equation and rearranging terms we obtain the equation of the circle $(x+1)^2 + y^2 = 4$. This circle is centered at the point $(-1, 0)$ and has radius 2. Since both $x+1$ and $y = \sqrt{4-(x+1)^2}$ are nonnegative, it follows that $\int_{-1}^{3} \sqrt{4-(x+1)^2} dx = 2\pi$ or quarter area of the circle.

4. Use Riemann sums to evaluate $\int_3^7 (4x+6) dx$ [10 pts]

Solution:
$\int_3^7 (4x+6) dx = \lim_{n\to\infty} \frac{4}{n} \sum_{k=1}^{n} \left[4\left(3 + k\frac{4}{n}\right) + 6\right] = \lim_{n\to\infty} \frac{4}{n} \sum_{k=1}^{n} \left(18 + k\frac{16}{n}\right)$.

Thus, $\int_3^7 (4x+6) dx = \lim_{n\to\infty} \frac{4}{n} \cdot 18n + \frac{4}{n} \cdot \frac{16}{n} \cdot \frac{n(n+1)}{2} = 4 \cdot 18 + 2 \cdot 16 = 4 \cdot 26 = 104$.

5. Compute $\lim_{n\to\infty} \frac{\pi}{2n}\left(\sin\left(\pi - 1\frac{\pi}{2n}\right) + \sin\left(\pi - 2\frac{\pi}{2n}\right) + \cdots + \sin\left(\pi - n\frac{\pi}{2n}\right)\right)$ [10 pts]

Solution: The above limit is a Riemann sum written from right to left of the integral $\int_{\pi/2}^{\pi} \sin\theta\, d\theta$. Thus the value of this limit is $\int_{\pi/2}^{\pi} \sin\theta\, d\theta = -\cos\pi = 1$.

6. Find $\frac{d}{dx}\int_x^{x^2} \sin t^2 \, dt$ [10 pts]

Solution: $\frac{d}{dx}\int_x^{x^2} \sin t^2 \, dt = 2x \sin x^4 - \sin x^2$

7. Compute $\int_{-1}^{1} \sin(\pi x^3) dx$. Be sure to justify your answer. [10 pts]

Solution: Let $f(x) = \sin(\pi x^3)$. Then $f(-x) = -f(x)$, which shows that $f(x)$ is an odd function. Since the interval of integration is centered at 0, we must have $\int_{-1}^{1} \sin(\pi x^3) dx = 0$

8. Calculate $\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x} dx$ [10 pts]

Solution: $\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x} dx = \int_{\pi/4}^{\pi/2} \cos x \sin^{-2} x \, dx = -\frac{1}{\sin x}\Big|_{\pi/4}^{\pi/2} = \sqrt{2} - 1$

9. Find a simple expression for $\int \frac{x}{\sqrt{4-9x^2}} dx$ [10 pts]

Solution: $\int \frac{x}{\sqrt{4-9x^2}} dx = \frac{-1}{18}\int \frac{-18x}{\sqrt{4-9x^2}} dx = \frac{-1}{9}\sqrt{4-9x^2} + C$

10. Calculate $\lim_{h\to 0} \frac{1}{h}\int_0^h f(x)dx$, where $f(x) = \begin{cases} \frac{\sin 2x}{x} & if\ x \neq 0 \\ 5 & if\ x = 0 \end{cases}$ [10 pts]

Solutions: $\lim_{h\to 0} \frac{1}{h}\int_0^h f(x)dx = \lim_{h\to 0} \frac{\sin 2h}{h} = 2$. What justifies this calculation?

Extra-Credit

11. Let $F(x) = \int_0^x t^2\, dt$ and $G(x) = \int_0^x x^2\, dx$. Is there any difference between the two functions? Justify your answer. [10 pts]

Solution: $F(x) = \int_0^x t^2\, dt$ represents the area under the curve $f(t) = t^2$ over the interval $[0, x]$, whereas $G(x) = \int_0^x x^2\, dx$, if at all meaningful, represents the area of the rectangle with base x and height x^2. Be sure to draw two pictures illustrating this!

12. Let $G(x) = \int_x^{\int_0^x v\, dv} \cos(t^2) dt$. Find $G'(x)$ [10 pts]

Solution: $G'(x) = \cos\left(\int_0^x v\, dv\right)^2 x = x\cos\left(\frac{1}{2}x^2\right)^2$

13. Show that $\int_a^b f\big(g(x)\big)g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$ [10 pts]

Solution: Let $F(x)$ be any anti-derivative of $f(x)$. Then

$$\int_a^b f\big(g(x)\big)g'(x)dx = F\big(g(x)\big)\Big|_a^b = F\big(g(b)\big) - F\big(g(a)\big) = F(u)\Big|_{g(a)}^{g(b)}$$
$$= \int_{g(a)}^{g(b)} f(u)du$$

14. Suppose that f is an even function with $\int_0^8 f(x)dx = 9$. Evaluate $\int_{-2}^{2} x^2 f(x^3)dx$. [10 pts]

Solution: By means of u-substitution we obtain

$$\int_{-2}^{2} x^2 f(x^3)dx = \frac{1}{3}\int_{-8}^{8} f(u)du$$

Since f is an even function, we can write

$$\frac{1}{3}\int_{-8}^{8} f(u)du = \frac{2}{3}\int_0^8 f(u)du = \frac{2}{3}\cdot 9 = 6$$